Notation and Mappings

Fact (Notation) — Three notations for the same function:

$$y = x^2 + 3 \qquad f(x) = x^2 + 3 \qquad f : x \mapsto x^2 + 3$$

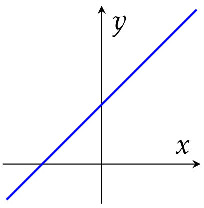
$f(2)$ means the output when the input is 2: $f(2) = 2^2 + 3 = 7$.

Example

$f(x) = x^2 + 3$. Find $f(-2)$, solve $f(x) = 28$, and simplify $f(2x)$.

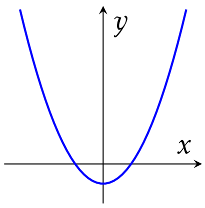
Definition. A **mapping** associates each value of an input set (the **domain**) with values of an output set (the **range**). There are four types:

one-to-one



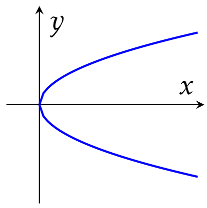
$y = 2x + 3$

many-to-one



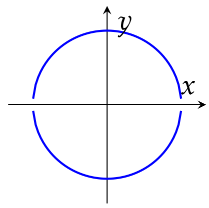
$y = 2x^2 - 1$

one-to-many



$y = \pm\sqrt{x}$

many-to-many



$x^2 + y^2 = 9$

Definition. A **function** is a mapping in which every input has exactly one well-defined output — i.e. a one-to-one or many-to-one mapping.

For example $y = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is not a function: the $\pm$ allows two outputs.

Example

Classify each mapping ($x \in \mathbb{R}$ unless stated), and state which are functions.

1. $x \mapsto x^3 + 1$
2. $x \mapsto x^2 - 4$
3. $x \mapsto \pm\sqrt{25 - x^2}, \quad -5 \le x \le 5$
4. $x \mapsto \cos x, \quad 0° \le x \le 360°$

Textbook Exercises: SPS Course 2.10, Exercise 1 Q1–11

Domain and Range

Fact — The domain lives on the x-axis, the range on the y-axis — both are sets of real numbers. State the domain in terms of x and the range in terms of y (or $f(x)$).

To find a range, sketch the graph.

Some inputs must be excluded for the output to exist:

Example

State the largest possible domain of:

1. $f(x) = \sqrt{6x-3}$
2. $g(x) = 2 + \dfrac{3}{2x-5}$
3. $h(x) = \dfrac{\sqrt{x+1}}{x-4}$

Example

Find the range of $f(x) = x^2 + 3x - 1, \ -1 \leq x \leq 1$.

Example

Find the range of $f(x) = (x-2)^2 + 3$ for each domain:

1. $x \in \mathbb{R}$
2. $0 \leq x \leq 4$
3. $1 \leq x \leq 5$

Example

Find the range of $f(x) = \dfrac{3x-1}{x-2},\ x \neq 2$.

Textbook Exercises: SPS Course 2.10, Exercise 1 Q12–13

Composite Functions

Definition. The composite function fg means "do g first, then f":

$$fg(x) = f(g(x)) \qquad x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$

The order is right-to-left. $f^2(x)$ means $ff(x)$ — not the same as $(f(x))^2$.

Example

$$f : x \mapsto 2x - 5 \qquad g : x \mapsto x^2 + 2 \qquad h : x \mapsto \frac{1}{x},\ x \neq 0$$

Find $ff(x)$, $fg(x)$, $gf(x)$ and $hgf(x)$. Why does the domain of hgf not exclude any value of x?

Note that $fg \neq gf$ in general.

Example

$f(x) = x^2$ and $g(x) = x + 2$. Solve $fg(x) = gf(x)$.

Example

$f(x) = 1 - \frac{1}{x}$, $x \neq 0, 1$. Find and simplify $f^2(x)$ and $f^3(x)$, and hence write down $f^{2026}(x)$.

Textbook Exercises: SPS Course 2.10, Exercises 2A and 2B

Inverse Functions

Definition. The inverse function f^{-1} returns each output of f to its input: $f^{-1}(f(x)) = x$.
f^{-1} exists only when f is **one-to-one** — otherwise some outputs would return to two inputs.

Tip (Finding an inverse)

Write $y = f(x)$; interchange x and y; make y the subject.

Fact — The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

Example

Find the inverse of $f(x) = \dfrac{1}{\sqrt[3]{x+2}}$.

A many-to-one function can be made one-to-one by restricting its domain.

Example

$f : x \mapsto x^2 - 6x + 16,\ \ x \geq k.$

1. Write f in the form $(x-a)^2 + b$.
2. State the smallest value of k for which f has an inverse.
3. With this k, find f^{-1}, stating its domain and range.

Example (Edexcel C3)

$$f : x \mapsto 1 - 2x^3,\ x \in \mathbb{R} \qquad g : x \mapsto \frac{3}{x} - 4,\ x > 0$$

Find f^{-1}.

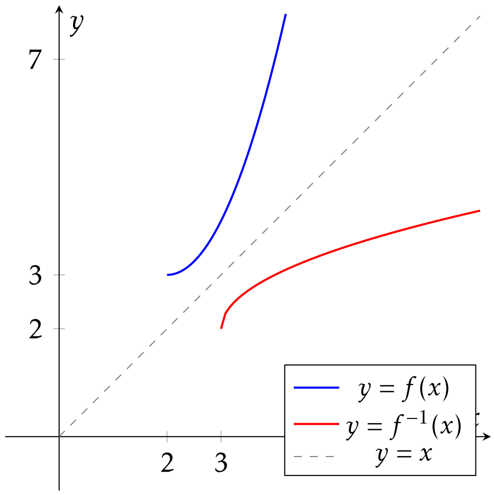
2. Show that $gf : x \mapsto \dfrac{8x^3 - 1}{1 - 2x^3}$.

3. Solve $gf(x) = 0$.

Textbook Exercises: SPS Course 2.10, Exercise 3A and Exercise 3B Q1–7

Graphs of Inverses

Fact — The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, because reflecting in $y = x$ interchanges x and y.



$f(x) = (x-2)^2 + 3,\ x \geq 2; \qquad f^{-1}(x) = 2 + \sqrt{x-3},\ x \geq 3$

Definition. f is self-inverse if $f^{-1} = f$, i.e. $ff(x) = x$ for all x in the domain. Its graph is symmetric about $y = x$.

Example

$$f(x) = \frac{2x-3}{x-2}, \quad x > 2$$

Find the range of f.

2. Show that $ff(x) = x$ for all $x > 2$.

3. Hence write down $f^{-1}(x)$.

Example (Edexcel C3)

$$f : x \mapsto 7x - 1,\ x \in \mathbb{R} \qquad g : x \mapsto \frac{4}{x-2},\ x \neq 2$$

Solve the equation $fg(x) = x$.

2. Hence find the largest value of a such that $g(a) = f^{-1}(a)$.

Remark. Solutions of $f(x) = f^{-1}(x)$ usually lie on $y = x$, but not always: a decreasing function can cross its inverse off the line.

Textbook Exercises: SPS Course 2.10, Exercise 3B Q8–20 and Revision Exercise 2.10